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Quantum field theory for non-polynomial chiral lagrangians

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Abstract. We deduce Green function equations and perturbation rules for chirally-invariant quantum field theory. The ordering problem is treated by a suitable time-like averaging in the operator interaction, which at the same time prevents any occurrence of $\delta^4(0)$ terms and removes ambiguities in $(\delta^3(0))^2$ terms; these latter are known to cancel to lowest order. We also present an equivalent lagrangian formulation which leads to the same results.

1. Introduction

There has been considerable progress recently in setting up quantum mechanics of chiral dynamics. The main problems in this program arise from the quadratic appearance of derivatives in the interaction term in the standard non-polynomial form of the classical chirally-invariant lagrangian (Barnes and Isham 1970):

$$\tfrac{1}{2}g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j. \tag{1.1}$$

Quantization of such a classical lagrangian presents many difficulties, ultraviolet and infrared divergences causing problems as well as the ordering and quadratically derivative interaction questions. However, it is necessary to set up the quantization program in as completely a consistent fashion as possible before we can properly attend to the divergence problem. This paper is devoted to that initial step. We will try to determine all the possible contributions which enter in Feynman diagrams, be they infinite or not. We will then give a brief description of how certain of the divergences discovered may be satisfactorily dealt with.

Let us start with the question of ordering. A hamiltonian approach has been followed by Charap (1973), who has shown that the requirement of chiral invariance determines the order of the non-commuting factors in the interaction completely and specifies the hamiltonian uniquely to within a unitary transformation. This uniqueness has been extended to the form of the energy–momentum tensor (Parish 1973) and to a possible set of Feynman rules (Davies 1973). An alternative approach to the ordering problem has been made by Bloore and his colleagues (Bloore and Routh 1973, Bloore *et al* 1973) using Frechet derivatives applied directly to the lagrangian. It is of interest to relate these two approaches, and furthermore develop Green function methods which allow perturbation theory rules to be explicitly derived.

We will first set up the equations for Green functions which follow from the symmetrized lagrangian of Charap. In the process we will find known terms proportional to $\delta^4(0)$ arising, along with those involving $(\delta^3(0))^2$. It is these latter terms in which we are most interested, for the ordering problem, especially since they must not appear in any finite predictions for S matrix elements. To discuss their disappearance in perturbation theory we will set up a functional differential equation for the generating functional of Green functions and solve it by functional Fourier transform techniques. It is at this stage that we can recognize an effective lagrangian. We then consider perturbation theory rules and the possible cancellation of terms proportional to $\delta^4(0)$ and $(\delta^3(0))^2$ in the effective lagrangian. The explicit removal of the $\delta^4(0)$ terms is described here using an averaging process which prevents any appearance of them, but that of the $(\delta^3(0))^2$ terms appears to involve much more of the dynamics than can be incorporated by our techniques. However, we show that our averaging process, necessary for a suitable Green function formulation of the properly-ordered field equations, leads to an unambiguous evaluation of $(\delta^3(0))^2$ contributions of particular Feynman diagrams. This leads to the expectation that all $\delta^4(0)$ and $(\delta^3(0))^2$ are completely cancelled in perturbation theory. The ground is thus basically cleared for discussion of ultraviolet and infrared divergences.

We then turn to the question of a lagrangian formulation of the theory, and show how the Frechet derivative method (Bloore *et al* 1973) allows for a complete derivation of the Green function and perturbation results from a lagrangian as compared to a hamiltonian starting point.

2. Green functional equations

We will take a four-dimensional chiral theory of mesons with symmetric hamiltonian

$$H = \int \mathrm{d}^3\boldsymbol{x}(\tfrac{1}{8}\{\phi_i\{p_j, g^{ij}\}\} + \tfrac{1}{2}\nabla\phi^i \cdot \nabla\phi^j g_{ij} + V) \tag{2.1}$$

where

$$V = \tfrac{1}{8}(\delta^3(0))^2 f_a{}^i{}_{,j} f_a{}^i{}_{,j} \tag{2.2}$$

$$[\phi^i(\boldsymbol{x}), p_j(\boldsymbol{y})]_- = \mathrm{i}\delta_{ij}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{2.3}$$

$$p_i = \tfrac{1}{2}\{g_{ij}, \dot{\phi}^j\} \tag{2.4}$$

and $\{A, B\} = AB+BA, [A, B] = AB-BA$.

We take the form of hamiltonian without the arbitrary function $h(\phi)$ of Charap (1973), though our discussion can easily be amended to include that term; we use the notation of Charap's paper. The extra term V allows H to be explicitly chirally invariant. The field equations which follow from (2.1) to (2.4) are

$$\dot{\phi}_i = \mathrm{i}[H, p_i] \tag{2.5}$$

or

$$\tfrac{1}{2}\{g_{ij}, \ddot{\phi}^j\} + \tfrac{1}{4}\{\{g_{ij,l}, \dot{\phi}^l\}\dot{\phi}^j\}$$
$$= -\tfrac{1}{2}\nabla\phi^l \cdot \nabla\phi^j g_{lj,i} - V_{,i} - \tfrac{1}{8}\{p_l\{p_j, g^{lj}{}_{,i}\}\} + \nabla \cdot (\nabla\phi^j g_{ij}) \tag{2.6}$$

where $V_{,i} = \partial V/\partial\phi^i$ and we use that for any suitably well-behaved function f of ϕ

$$\mathrm{d}f/\mathrm{d}t = \dot{f} = \tfrac{1}{2}\{f_{,i}, \dot{\phi}^i\} \tag{2.7}$$

(as can be easily obtained by expanding $f(\phi)$ in a power series in its variable). We rewrite (2.6) in a form involving only ϕ and its derivatives by means of (2.4):

$$\tfrac{1}{2}\{g_{ij}, \Box\phi^i\} + \tfrac{1}{4}\{\{\Gamma_{ijk}, \partial_\mu\phi^k\}\partial^\mu\phi^j\} + W_i = 0 \tag{2.8}$$

with

$$\Gamma_{ijk} = g_{ij,k} - \tfrac{1}{2} g_{jk,i} \tag{2.9}$$

where we use the canonical commutation relations and the consequent relation

$$[\phi^i(\boldsymbol{x}), \dot{\phi}^j(\boldsymbol{y})]_- = -\mathrm{i} g^{ij} \delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{2.10}$$

to derive for W_i the value

$$W_i = V_{,i} - \tfrac{1}{8}(\delta^3(0))^2 g^{lk}{}_{,m}(g^{jm} g_{jk,i})_{,l}. \tag{2.11}$$

We note here the difference between the expression for Γ_{ijk} given by (2.9) and its usual value $\frac{1}{2}(g_{ij,k}+g_{ik,j}-g_{jk,i})$, the difference only arising in the time derivatives due to the lack of commutativity between the various components $\dot{\phi}^l$ for different l.

From the field equation (2.8) and the commutation relation (2.10) we may attempt to derive the Green functions equations by the usual techniques. We do this by means of the generating functional $G(J)$ of the Green functions, defined as usual by

$$G(J) = \langle 0 | T[\exp(\mathrm{i} \textstyle\int J_\alpha(x) \phi^\alpha(x) \, \mathrm{d}^4 x)] | 0 \rangle . \tag{2.12}$$

The Green functions $\tau^{\alpha_1 \ldots \alpha_n}(x_1 \ldots x_n) = \langle 0 | T(\phi^{\alpha_1}(x_1) \ldots \phi^{\alpha_n}(x_n)) | 0 \rangle$ which are generated by $G(J)$ are assumed to be well defined, though we cannot expect that is the case if time-ordered products of the non-commuting fields ϕ and $\dot{\phi}$ at the same time values are involved. We may avoid the resulting ambiguity by a symmetrization procedure. Thus if we consider in particular the ambiguous time-ordered product

$$-\mathrm{i} g_{ij}(-\mathrm{i}\delta/\delta J(x)) \Box \, \delta/\delta J_j(x) G(J) = \langle 0 | T[g_{ij} \Box \, \phi^j \exp(\mathrm{i} \textstyle\int J\phi \, \mathrm{d}^4 y)] | 0 \rangle \tag{2.13}$$

the ambiguity of the ordering on the right-hand side of (2.13) can be removed if we define the left-hand side by a limiting procedure thus:

$$\lim_{\epsilon \to 0} -\mathrm{i} g_{ij}(-\mathrm{i}\delta/\delta J(x)) \Box \, \delta/\delta J_j(x+\epsilon) G(J) = \langle 0 | T[\tfrac{1}{2}\{g_{ij}(x), \Box \phi^j(x)\} \exp(\mathrm{i} \textstyle\int J\phi \, \mathrm{d}^4 y)] | 0 \rangle \tag{2.14}$$

where the symbol $\lim_{\epsilon \to 0}$ denotes an average over time-like infinitesimal four-vectors ϵ, and we have neglected the $\delta^4(0)$ term arising on the right of (2.14) for the moment.

We will use the time-like average over ϵ from now on, though not explicitly indicate it, so that when formulae with ϵ appear, such an averaging process is to be taken at the end of all other operations.

We note that the right-hand side of (2.14) is still not well defined due to the non-commutativity of $\Box\phi$ with the exponential involving ϕ at coincident points; the time-ordering symbol T has not been defined to take account of such an ambiguity. However, a similar difficulty arises even in quantum field theories without derivative coupling; the situation is saved by the use of the field equations to replace the troublesome $\Box\phi$ by a polynomial in the fields ϕ. These commute with other fields entering at the same point, so removing any ambiguity. We will use the same approach here.

We now wish to use the field equation (2.8) to derive a functional differential equation for $G(J)$. To take account of the ordering in (2.8) correctly, especially that of the double commutator bracket of the second term, we will have to extend the symmetrization procedure of (2.14). Let us first define the time-like averaged limit of the left-hand side of (2.14) in more detail. Thus we define, for general non-commuting operators f, g inside a time-ordered product:

$$\lim_{\epsilon\to 0} f(x)g(x+\epsilon) = \lim_{\epsilon\to 0+} \tfrac{1}{2} \sum_{\lambda=-1}^{+1} f(x)g(x+\omega(\lambda,\epsilon)) \tag{2.15}$$

where

$$\omega(\lambda,\epsilon) = (\lambda\epsilon, \mathbf{0}).$$

If f and g are smooth in their variables, in some sense, we expect (2.15) to give the symmetrized product $\frac{1}{2}\{f(x), g(x)\}$. We extend this to the case of three non-commuting operators f, g, h inside a time-ordered product. Thus

$$\lim_{\epsilon,\epsilon'\to 0} f(x)g(x+\epsilon)h(x+\epsilon') = \lim_{\substack{\epsilon,\epsilon'\to 0+\\ \epsilon'>\epsilon}} \sum_{\lambda,\lambda'=-1}^{+1} f(x)g(x+\omega(\lambda,\epsilon))h(x+\omega(\lambda',\epsilon')). \tag{2.16}$$

We have chosen the range of integration on the right-hand side of (2.16) so that for reasonably smooth functions f, g and h the right-hand side of (2.16) becomes

$$\tfrac{1}{4}\{\{f(x), g(x)\}h(x)\}.$$

We may now use the field equation (2.8) and the commutator (2.10) to obtain the Green functional equation

$$\lim_{\epsilon,\epsilon'\to 0} \{-\mathrm{i}g_{ij}(-\mathrm{i}\delta/\delta J(x))\,\Box\,\delta/\delta J(x+\epsilon) + W_i(-\mathrm{i}\delta/\delta J(x)) - J_i(x+\epsilon)$$
$$-\Gamma_{ijk}(-\mathrm{i}\delta/\delta J(x))\partial_\mu\delta/\delta J_j(x+\epsilon)\partial^\mu\delta/\delta J_k(x+\epsilon')\}G(J) = 0. \tag{2.17}$$

Naturally the double limit procedure of (2.16) is only applicable to the second term in (2.17). Nor do we obtain the usual term containing $\delta^4(0)$ in (2.17), since the two time derivatives in the second term of (2.17) are evaluated at *different* points $x+\omega(\lambda,\epsilon)$, $x+\omega(\lambda',\epsilon')$. If we had allowed $\epsilon = \epsilon'$ then this singular term, actually equal to

$$\tfrac{1}{2}\mathrm{i}\delta^4(0)\,\mathrm{Tr}\,\ln g(-\mathrm{i}\delta/\delta J(x))_{,i}G(J) \tag{2.18}$$

would have to be added to (2.17). This term has in any case been shown to cancel to all orders with singular parts of perturbation graphs (Keck 1971), but there are very specific terms left behind. We will have to show that the absence of the explicit appearance of the $\delta^4(0)$ term in our equation (2.17) still allows for the complete and correct removal of the further terms containing $\delta^4(0)$ generated by perturbation theory; we will only be able to do that once we have obtained the full perturbation theory rules so that these further terms can be obtained explicitly.

Before we do that we note that the ambiguity of ordering still present on the right-hand side of (2.14) is removed in (2.17) provided the first and fourth terms are treated together, giving an unambiguous operator equal to the third and second terms, by the field equations. Thus we must derive perturbation rules in which the first and fourth terms are automatically combined together. This we now do.

3. The functional integral solution

To proceed towards perturbation rules it is useful to solve (2.17) by a formal functional Fourier transform:

$$G(J) = \int \prod_{x,j} \mathrm{d}K^j(x) \exp(\mathrm{i}\textstyle\int J_j(x)K^j(x)\,\mathrm{d}^4x)\tilde{G}(K) \tag{3.1}$$

so that (2.17) becomes

$$\lim_{\epsilon,\epsilon'\to 0} [g_{ij}(K(n)) \Box K^j(x+\epsilon) - \Gamma_{ijk}(K(n))\partial_\mu K^j(x+\epsilon)\partial^\mu K^k(x+\epsilon') + W_i(K(n)) + \mathrm{i}\delta/\delta K^j(x+\epsilon)]\tilde{G}(K) = 0 \tag{3.2}$$

with solution, to within a normalization factor, which is

$$\tilde{G}(K,\epsilon,\epsilon') = \exp\left(-\mathrm{i}\int \mathrm{d}^4x[\tfrac{1}{2}\partial_\mu K^i(x+\epsilon)g_{ij}(K(n))\partial^\mu K^j(x+\epsilon') - W(K(n))]\right). \tag{3.3}$$

That (3.3) does actually satisfy (3.2) in the appropriate limits $\epsilon, \epsilon' \to 0$ is shown by explicit calculation, using the rules (2.15) and (2.16) and that

$$\partial_\mu g_{ij}(K(n))\tilde{G}(K) = \tfrac{1}{2}\{g_{ij,l}, \partial_\mu K^l\}\tilde{G}(K) = \lim_{\epsilon\to 0} g_{ij,l}(K(n))\partial_\mu K^l(x+\epsilon)\tilde{G}(K)$$

together with the operator form of $\hat{G}$:

$$\tilde{G}(K) = \left\langle 0 \left| T\left(\prod_{x,l} \delta(K^l(x) - \phi^l(x))\right)\right| 0 \right\rangle.$$

the meaning given to the δ function in the right-hand side of this latter expression being by Taylor expansion in powers of the operator ϕ^l. What results from such a calculation is that $G(K,\epsilon,\epsilon')$ of (3.3) satisfies a functional differential equation which, for the limits of ϵ and ϵ' tending to zero as defined by (2.15) and (2.16), is identical to (2.17). We thus consider $\tilde{G}(K)$, defined by

$$\tilde{G}(K) = \lim_{\epsilon,\epsilon'\to 0} \tilde{G}(K,\epsilon,\epsilon') \tag{3.4}$$

as the appropriate Green functional for the chiral field theory with hamiltonian (2.1). The limits in (3.4) are only to be taken at the very end of any calculation of Green functions.

We may regard these Green functions arising from the effective operator lagrangian

$$L = \tfrac{1}{8}\{\{g_{ij}, \partial_\mu\phi^i\}\partial^\mu\phi^j\} - W \tag{3.5}$$

where $W_{,i} = W_i$, with W being well defined. We have not shown that if we started from L of (3.5), now regarded as an operator lagrangian, we would obtain (2.8) as the Euler–Lagrange variational equations. To show that we will have to discuss non-commuting variations; we return to this question later. But first we will consider the functional integral solution (3.1), (3.3), (3.4) more completely so as to put it into a form where we can analyse the terms containing $\delta^4(0)$ in a more explicit fashion. At the same time we will obtain an expression for the S matrix from which the perturbation rules may be read off directly.

To proceed we write

$$L = \tfrac{1}{2}\partial_\mu K^i\delta_{ij}\partial^\mu K^j + L_{\text{int}}$$

with

$$L_{\text{int}} = \tfrac{1}{2}\partial_\mu K^i(x+\epsilon)[g_{ij}(K(n)) - \delta_{ij}]\partial_\mu K^j(x+\epsilon') - W(K(n)). \tag{3.6}$$

Then, as usual (Fried 1973)

$$G(J) = \exp(\mathrm{i}\textstyle\int L_{\text{int}}(-\mathrm{i}\delta/\delta J(x))\,\mathrm{d}^4x)\exp(\tfrac{1}{2}\mathrm{i}JDJ) \tag{3.7}$$

where

$$JDJ = \int J^i(x)D(x-y)J^i(y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.8}$$

and $\Box D(x) = \delta^4(x)$.

We can put this into a form more amenable to determining the required terms if we rewrite (3.7), using the rearrangement formula (Fried 1973) extended to derivatives,

$$\exp[\tfrac{1}{2}\mathrm{i}(JDJ)]\exp(\tfrac{1}{2}\mathrm{i}\delta/\delta A^i_{,\mu}D_{,\mu\nu}\delta/\delta A^i_{,\nu} - \mathrm{i}\delta/\delta A^i_{,\mu}D_{,\mu}\delta/\delta A^i$$
$$-\tfrac{1}{2}\mathrm{i}\delta/\delta A^i D\delta/\delta A^i)\exp(\mathrm{i}\textstyle\int A^i_{,\mu}f_{ij}A^j_{,\mu}) \tag{3.9}$$

where

$$f_{ij} = g_{ij} - \delta_{ij}, \qquad \Box A^i = J^i, \qquad A^i_{\ ,\mu} = \partial A^i/\partial x^\mu,$$
$$D_{,\mu} = \partial D/\partial x^\mu, \qquad D_{,\mu\nu} = \partial^2 D/\partial x^\mu \partial x^\nu.$$

We understand in (3.9) that A and $A_{,\mu}$ are to be regarded as independent variables until all the functional derivatives have been performed as indicated. We may then replace $A_{,\mu}$ by $\partial_\mu A/\partial x^\mu$ evaluated at $x+\epsilon$, $x+\epsilon'$ respectively (in the last factor in (3.9)), the time-like average over ϵ, ϵ' being taken finally in the manner prescribed in (2.15) and (2.16).

We perform the derivations with respect to $A_{,\mu}$ explicitly in (3.9) to give (Fried 1973)

$$G(A) = \exp(\tfrac{1}{2}\mathrm{i}JDJ)\exp(-\tfrac{1}{2}\mathrm{i}\delta/\delta A^i D\delta/\delta A^i)\exp[\tfrac{1}{2}\mathrm{i}(A_{,\mu} - \mathrm{i}D_{,\mu}\delta/\delta A)f(1+D_{,\dots}f)^{-1}{}_{\mu\nu}$$
$$\times (A_{,\nu} - \mathrm{i}D_{,\nu}\delta/\delta A) - \tfrac{1}{2}\,\mathrm{Tr}\ln(1+D_{,\dots}f) - \mathrm{i}W(A)]. \tag{3.10}$$

The explicit ϵ, ϵ' dependence in (3.10) has been suppressed, but is given in detail by replacing $f(x)$ throughout (3.10) by the kernel F_{xy} given by

$$F_{xy} = \int \mathrm{d}^4 z f(z)\delta^4(x-z-\epsilon)\delta^4(y-z-\epsilon'). \tag{3.11}$$

Further we note that the excluded $\delta^4(0)$ term would have given the contribution

$$\tfrac{1}{2}\delta^4(0)\ln\det g(A)$$

in the last exponential on the right-hand side of (3.10). The chiral indices on the variables $A_{,\mu}$, f have been dropped for simplicity in the term in (3.10) but should be understood as being present implicitly from now on.

The formula (3.10) allows a direct perturbation interpretation in the usual manner, the first factor on the right-hand side being the set of free-particle contributions which are modified by the action of the second factor operating on the third. An interpretation of this latter has already been given, in terms of the supergraphs arising from the expansion of the third exponential in (3.10) in a power series (Keck and Taylor 1971). The supergraphs are those expected from the classical chiral lagrangian (1.1) each supergraph of N vertices having the interaction part f_{ij} assigned to the vertex along with a pair of double lines, these latter corresponding either to an external line coming from $A_{,\mu}$ or an internal one from an end of $D_{,\mu\nu}$. The double lines will form continuous non-intersecting curves from the expansion of the denominator in the third line of (3.10), each line ending externally with $A_{,\mu}$ or being attached through a half-double line $D_{,\mu}$ to another vertex. There will also be closed loops of double lines arising from the Tr ln term in (3.10), along with other vertices each comprising the term proportional to $\delta^4(0)$ (if it had been allowed) or the additional potential term $W(A)$ containing $(\delta^3(0))^2$. The second factor in (3.10) will cause single lines to join all of these vertices in all possible ways as is usual in perturbation theory for non-polynomial lagrangian theories. The ϵ, ϵ' dependence is made explicit in these perturbation rules if we take the double line contribution from x to y to be given by

$$D_{,\mu\nu}(x+\epsilon'-y-\epsilon) \tag{3.12}$$

as follows immediately from (3.11).

The resulting perturbation rules do not seem to agree with those of Davies (1973) due to omission of all the terms but V_i in W_i. Our approach contains no ambiguities; we can only conclude that the extra terms we have found here arise by a more careful treatment in Davies' discussion along the lines of Gerstein *et al* (1971).

In detail our perturbation diagrams thus have extra vertices, beyond those of Davies, due to the non-polynomial non-derivative interaction term in the lagrangian equal to $U(K)$, where

$$U_{,i}(K) = \tfrac{1}{8}(\delta^3(0))^2 g^{lk}{}_{,m}(g^{jm}g_{jk,i})_{,l}. \tag{3.13}$$

This term has arisen due to the transformation from the fields K^i and momenta p_i of § 2 to the fields K^i and their time derivatives; this extra term is apparent in comparison of (2.1) and (2.8). This change of variables cannot be as straightforward as is claimed by Davies, who neglected any problem of ordering in making it to obtain his perturbation theory rules.

This can be seen in detail in setting up the usual transformation $U(t, t_0)$ from p_i to $\dot{K}^i$; in the interaction representation the relevant equation, in terms of U, is equation (2.11) of Davies (1973) which becomes

$$U(\hat{p}_i+\tfrac{1}{2}[\hat{g}_{ij}, \hat{p}_j]_+)U^{-1}=\tfrac{1}{2}[g_{ij}, \dot{K}_j]_+ +\tfrac{1}{16}U[\hat{g}_{ij}, [\hat{F}^{aj}, [\hat{p}_k, \hat{F}^{ak}]_+]_+]_+U^{-1} \tag{3.14}$$

where the carets denote Heisenberg picture operators. This evidently has the solution $U\hat{p}_iU^{-1} = \dot{K}_i$ if the ordering in the second term on the right-hand side of (3.14) is neglected. Correct account of the ordering produces the term U given by (3.13).

4. Cancellations

The quartic divergences arising from the closed double line loop factor are known to cancel the term proportional to $\delta^4(0)$ exactly (Keck 1971), provided this term arises. Our analysis of the previous two sections has not required any such contributions. To show that the terms involving $\delta^4(0)$ generated by the perturbation rules are also absent in our approach it may be helpful to give here the argument for the complete cancellation of the $\delta^4(0)$ terms. So we will use (3.10) with ϵ and ϵ' set zero for the present. We can write the vacuum double loop term in (3.10) as

$$-\tfrac{1}{2}\operatorname{Tr}\ln(1+D_{,\dots}f) = -\tfrac{1}{2}\sum_{n\geqslant 1}\frac{1}{n}\operatorname{Tr}[(D_{,\dots}f)^n]. \tag{4.1}$$

We evaluate the divergences in each term on the right-hand side of (4.1) by Fourier transformation:

$$\overbrace{D_{,\mu_1}{}^{\mu_2}D_{,\mu_2}{}^{\mu_3}\dots D_{,\mu_n}{}^{\mu_1}(p_1\dots p_n)} = \int\prod_{j=1}^{n}\frac{\mathrm{d}x_j}{(2\pi)^4}\mathrm{e}^{\mathrm{i}p_jx_j}\frac{\mathrm{d}q_j}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}q_j(x_{j+1}-x_j)}\frac{(q_{j+1}, q_j)}{q_j^2} \tag{4.2}$$

where $x_1 = x_{n+1}$ and $q_1 = q_{n+1}$, $q_0 = q_n$. Then $q_{r+1} = q_r + p_{r+1}$, and if we use the variable $q = q_n$ then (4.2) becomes

$$(2\pi)^{-4(n-1)} \int dq \prod_{r=1}^{n} \left\{ \left(q + \sum_{j=1}^{r+1} p_j, q + \sum_{j=1}^{r} p_j \right) \left[\left(q + \sum_{j=1}^{r} p_j \right)^2 \right]^{-1} \right\} \delta^4 \left(\sum_{j=1}^{r} p_j \right). \tag{4.3}$$

Expressing the numerator of (4.3) in terms of $P_r = \Sigma_{j=1}^{r} p_j$ and then the integrand of (4.3) as a power series expansion in inverse powers of q gives for (4.2) the expression

$$\frac{\delta^4(\Sigma_{j=1}^{r} p_j)}{(2\pi)^{4(n-1)}} \int d^4q \left[1 + \sum_{r=1}^{n} \frac{(P_r, p_{r+1})}{q^2} - \sum_{r=1}^{n} \frac{2(q, p_{r+1})(q, P_r)}{(q^2)^2} + \sum_{r,s=1}^{n} \frac{(q, p_{r+1})(q, p_{s+1})}{(q^2)^2} + O\left(\frac{1}{(q^2)^2} \right) \right]. \tag{4.4}$$

If we neglect all except the quartic divergence in (4.4) then

$$D_{,\mu_1}{}^{\mu_2}(x_1 - x_2) D_{,\mu_2}{}^{\mu_3}(x_2 - x_3) \ldots D_{,\mu_n}{}^{\mu_1}(x_n - x_1) = \delta^4(0) \prod_{j=2}^{n} \delta^4(x_1 - x_j) \tag{4.5}$$

where we have defined $\delta^4(0) = (2\pi)^{-4} \int d^4q$. Thus the contribution of the quartic divergence from the closed double line loops will be

$$-\tfrac{1}{2}\delta^4(0) \sum_{n \geqslant 1} \frac{1}{n} \int \mathrm{Tr}\, f^n(x)\, d^4x = -\tfrac{1}{2}\delta^4(0)\, \mathrm{Tr} \ln g. \tag{4.6}$$

This is seen to cancel identically the term explicitly proportional to $\delta^4(0)$ which we excluded from (3.10). We can also evaluate the quadratically and even logarithmically divergent terms of (4.1) from (4.4) on inclusion of the $O(q^{-4})$ term. However, their cancellation is not obvious, and we will turn now to the cancellation of the term (4.6) in our more careful analysis of (3.10) keeping the limiting procedure of ϵ and ϵ' tending to zero.

If we include the ϵ and ϵ' dependence of (4.1) then we need to include on the right-hand side of (4.2) the additional factor

$$\exp\left(i(\epsilon - \epsilon') \sum_{j=1}^{n} q_j \right). \tag{4.7}$$

This produces the extra factor in (4.3) equal to

$$\exp\{ i(\epsilon - \epsilon')[nq + np_1 + (n-1)p_2 + \ldots + p_n] \}. \tag{4.8}$$

Using the expansion of (4.4) the quartic divergence of the unambiguous form of (4.1) is

$$-\tfrac{1}{2}\delta^4(\epsilon - \epsilon')\, \mathrm{Tr} \ln g$$

which is zero as $\epsilon, \epsilon' \to 0$ if the limiting prescription of § 2 is used, with $\epsilon \neq \epsilon'$. Thus there is no term proportional to $\delta^4(0)$ in the perturbation expansion for the Green functions, either arising explicitly from the quadratic derivative interaction or from closed loops of twice-differentiated propagators.

We are still left with the term W in (3.10) which is proportional to $(\delta^3(0))^2$.

The question of cancellation of this term has already been discussed in the lowest non-trivial order (Suzuki and Hattori 1972), where a complete cancellation was obtained. We would like to show such a cancellation, but now extended to terms of all order, possibly along the lines of the preceding removal of the $\delta^4(0)$ term. The discussion of the lowest order perturbation theory generation of the $(\delta^3(0))^2$ term shows that such a term arises from singular products of the form

$$D_{,\mu\nu}(x)D_{,\mu\nu}(x)D(x). \tag{4.9}$$

This term was evaluated by Suzuki and Hattori (1972) by using the correct expansion

$$\partial_\mu\partial_\nu D = -\delta_{\mu 0}\delta_{\nu 0}\delta^4(x)+\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\omega(\boldsymbol{k})}k_\mu k_\nu(\theta(x_0)\,\mathrm{e}^{-ikx}+\theta(-x_0)\,\mathrm{e}^{ikx}). \tag{4.10}$$

The product of (4.9) involves the ambiguous product $\theta(x_0)\delta(x_0)$, which was taken to be $\frac{1}{2}\delta(x_0)$ by Suzuki and Hattori without very strong grounds. The presence of ϵ and ϵ' should allow us to obtain an unambiguous value for this product. We have to calculate

$$D_{,\mu\nu}(x+\epsilon-\epsilon')D_{,\mu\nu}(x+\epsilon'-\epsilon)D(x). \tag{4.11}$$

When this product is evaluated by (4.10) the crucial factors to be averaged over suitable time-like directions of ϵ and ϵ' according to (2.16) are $\theta(\epsilon_0'-\epsilon_0)$ or $\theta(\epsilon_0-\epsilon_0')$; it is almost immediate that these each average to give the factor $\frac{1}{2}$; so agreeing with Suzuki and Hattori but also giving a more careful analysis of the reasons for this result. By their analysis there is cancellation between the $(\delta^3(0))^2$ contribution in (4.11) and that arising from the term of appropriate order in W.

We can see this most clearly from the use of Weinberg coordinates, for which

$$g_{ij} = \delta_{ij}\Phi^{-1}, \qquad g^{ij} = \delta_{ij}\Phi, \qquad \Phi = (4\lambda^2\phi^2)^2 \tag{4.12}$$

for which

$$W = -\tfrac{1}{2}(\delta^3(0))^2\lambda^4\phi^2+\mathrm{O}(\lambda^6) \tag{4.13}$$

on dropping a constant term. But we see from (3.4) that the term W is just the same as that resulting from the calculation of Suzuki and Hattori for the $(\delta^3(0))^2$ term in their effective hamiltonian (4.18) (Suzuki and Hattori 1972). Since the $(\delta^3(0))^2$ contribution from the perturbation graphs due to twice-differentiated propagators are to be calculated by exactly the symmetrization prescription $\theta(x_0)\delta(x_0) = \frac{1}{2}\delta(x_0)$ of Suzuki and Hattori then the cancellations of the $(\delta^3(0))^2$ terms found in that paper are preserved.

The difficulty in evaluating all such $(\delta^3(0))^2$ terms completely is that they apparently involve a great deal of the detailed dynamics not made explicit in the expression (3.10). Removal or prevention of the $\delta^4(0)$ terms only required discussion of closed-loops of double lines. It is necessary to go beyond that to include the non-derivative terms to obtain the coefficients of the various terms like (4.11). This appears to require almost a complete solution of the perturbation expansion, and so need a great deal of further work. Thus while we expect an exact cancellation of all of the $(\delta^3(0))^2$ terms in the final expression for S matrix elements we cannot show that occurs by the methods developed here.

5. Lagrange equations

The canonical quantization procedure is the standard one to apply to a classical field theory, starting from a lagrangian $L(\phi, \partial_\mu\phi)$. The Euler–Lagrange variational equations are then the quantum field equations, the operator structure being ensured by the canonical commutation or anticommutation relations, between the fields ϕ and their canonical momenta $\partial L/\partial\dot{\phi}$. There is a crucial difficulty in this approach when L involves a quadratically derivative interaction, since then the variational equation is not well defined. This problem has recently been discussed by Bloore and his co-workers (Bloore *et al* 1973, Bloore and Routh 1973) who used the Frechet derivatives, for a special class of variations, in order to derive equations of motion consistent with the symmetry present in the lagrangian.

The lagrangian approach is of value in that it leads rapidly to perturbation rules which are explicitly covariant. This is certainly true for the chiral interaction considered here, though our derivation of the Green function equations in § 2 soon removed this blemish. But in a theory in which the symmetry group causes great difficulty in the hamiltonian approach, as for example in general relativity, it appears advantageous to construct a quantization procedure which uses purely the lagrangian formulation. We will now attempt to develop such an approach for the chiral theory considered so far in this paper.

Our main problem is that the quantum lagrangian L associated with the classical hamiltonian (2.1) (with the term V removed) is not uniquely defined due to the ordering problem. We attempt to resolve this ambiguity by requiring invariance of L under the class of Killing vector displacements:

$$\delta\phi^i = f^i_a\theta^a \tag{5.1}$$

and the associated displacement of the derivatives

$$\delta\partial_\mu\phi^i = \tfrac{1}{2}\{f^i_{a,j}, \phi^j{}_{,\mu}\}\theta^a + f^i_a\partial_\mu\theta^a \tag{5.2}$$

where f^i_a are the Killing vectors associated with the metric g_{ij} and depend only on the field components ϕ^l at the space–time point, while the θ^a are arbitrary c number functions of space–time.

We take the lagrangian, from (3.5), to be

$$L = \tfrac{1}{8}\{\{g_{ij}, \dot\phi^i\}\dot\phi^j\} - \tfrac{1}{2}g_{ij}\nabla\phi^i \,.\, \nabla\phi^j - W. \tag{5.3}$$

The equations of motion which arise from the variational equation $\delta L = 0$ for (5.3), under the variations (5.1) and (5.2), are

$$\int \mathrm{d}^4x\left[\left(\frac{\partial L}{\partial\phi^k}, f^k_a\theta^a\right) + \left(\frac{\partial L}{\partial\partial_\mu\phi^k}, \partial_\mu(f^k_a\theta^a)\right)\right] = 0 \tag{5.4}$$

where the bracket expression $(\partial L/\partial q, b)$ is the result of the operation of the Frechet derivative $\partial L/\partial q$ acting on the operator b, as used by Bloore and collaborators (Bloore and Routh 1973, Bloore *et al* 1973). An integration by parts allows us to rewrite (5.4) as the usual Euler–Lagrange equations

$$\frac{\partial L}{\partial\phi^k} - \partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\phi^k}\right) = 0. \tag{5.5}$$

We may use the fact that $\partial L/\partial\dot\phi^k$ is the momentum p_k of (2.4) and the equation of motion (2.5), (2.6) and (2.8) to replace the second term on the left-hand side of (5.5) by

$$\tfrac{1}{8}\{\{g_{ij,k}, \partial_\mu\phi^i\}\partial^\mu\phi^j\} - W_k$$

which is equal to the first term in (5.5) by direct computation. Thus the lagrangian (5.3) gives variational equations of motion for the fields agreeing with the hamiltonian equations of motion. This is in disagreement with the remarks of Charap (1973), but we are using here that the only vector upon which W_k of (2.11) can depend is ϕ^k, so that there is some function $f(\phi^2)$ for which

$$W_k(\phi) = f(\phi^2)\phi^k. \tag{5.6}$$

Thus the consistency condition for integrability of W_k

$$W_{k,l} = W_{l,k}$$

is automatically satisfied, and

$$W_k = W_{,k}, \qquad W = h(\phi^2) = \tfrac{1}{2}\int_0^{\phi^2} f(x)\,\mathrm{d}x. \tag{5.7}$$

We further remark that the form (5.3) of the lagrangian gives an action which automatically possesses the symmetry of the theory under the transformations (5.1) and (5.2) since constant θ^a cause equation (5.4) to reduce exactly to the invariance condition. Finally we note that the perturbation rules will be exactly the same as before, the operator ordering in (5.3) having to be handled as before, so leading to total cancellation of $\delta^4(0)$ terms and unambiguous evaluation and cancellation of lowest order $(\delta^3(0))^2$ terms.

It is possible to start from the lagrangian, of (5.3), chosen as the unique form possessing invariance under (5.1) and (5.2), and deduce the hamiltonian (2.1) by requiring it to give the same equation of motion as (5.5). This lagrangian approach may be of great help in discussing the ordering problem in such theories as quantized gravity; we hope to discuss this elsewhere.

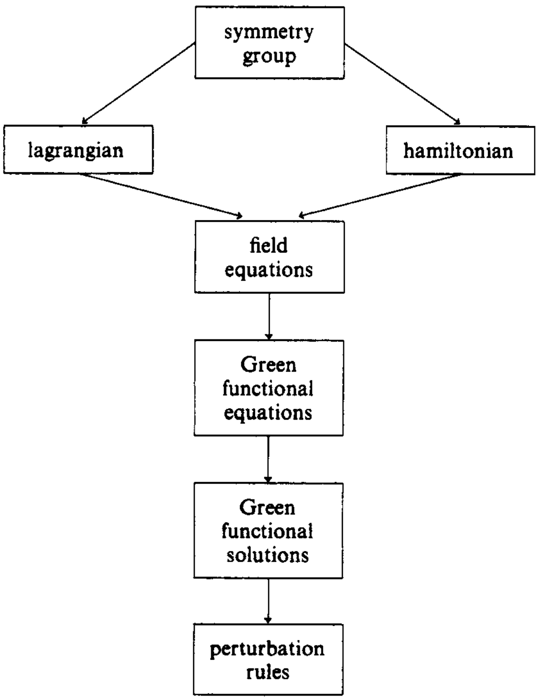

Figure 1. Flow chart showing the logical connection between the various steps in deriving perturbation rules for a chirally invariant quantum field theory.

We finally present, in figure 1, a flow chart of the various methods of approach to a unique perturbation theory; from the symmetry group we derive equivalent hamiltonian and lagrangian schemes which lead to perturbation rules by the methods we have outlined in the previous sections.

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